INHOMOGENEOUS DIOPHANTINE APPROXIMATION AND ANGULAR RECURRENCE FOR POLYGONAL BILLIARDS

JÖRG SCHMELING AND SERGE TROUBETZKOY

Abstract. For a given rotation number we compute the Hausdorff dimension of the set of well approximable numbers. We use this result and an inhomogeneous version of Jarnik's theorem to show strong recurrence properties of the billiard flow in certain polygons.

1. Introduction

In the past decade in four independent articles it was observed that the billiard orbit of any point which begins perpendicular to a side of a polygon and at a later instance hits some side perpendicularly retraces its path infinitely often in both senses between the two perpendicular collisions and thus is periodic. The earliest of these articles is a numerical work of Ruijgrok which conjectures that every triangle has perpendicular periodic orbits [R]. In 1992 Boshernitzan [B] and independently Galperin, Stepin and Vorobets [GSV] proved that for any rational polygon, for every side of the polygon, the billiard orbit which begins perpendicular to that side is periodic for all but finitely many starting points on the side. Finally for an irrational right triangle Cipra, Hansen and Kolan have considered points which are perpendicular to one of the legs of the triangle. They showed that for almost every such point the billiard orbit is periodic [CHK]. Here the almost everywhere statement is with respect to the length measure on the side considered. The method of their proof in fact implies a stronger result, namely that in any (generalized) parallelogram or right triangle, for each direction θ, the set of points F_θ whose orbit starts in the direction θ and never returns parallel to itself has Lebesgue measure 0 [GT]. The result of Cipra *et. al.* on periodic orbits follows from the above observation. Their computer simulation indicated that perhaps one can improve the almost everywhere and that is the starting point of this research. In Theorem 4.1 we show that in any (generalized) parallelogram, for each direction θ, the set of points whose orbit starts in the direction θ and never returns parallel to itself has lower box dimension at most one half. A corollary to this theorem is that the set of points whose orbit starts perpendicular to a leg of a right triangle whose orbit is not periodic has lower box dimension at most one half.

We then turn to the question, whether there are directions θ for which we can improve the constant $1/2$ in Theorem 4.1. In Theorem 4.4 we show that if $\mu > 1$ is the approximation order of θ by α, then in fact the lower box dimension of F_θ is at most $(\mu+1)^{-1}$. We have two applications of Theorem 4.4. We consider the set $\mathcal{C}_s$ of directions θ for which the set of points which never return parallel to themselves have lower box dimension at most $s \in [0, 1/2]$. In Theorem 4.6 we prove that the Hausdorff dimension of $\mathcal{C}_s$ is at least $s/(1-s)$, the set $\mathcal{C}_s$ is residual and has box dimension 1. Fix a direction θ_0 and consider the set $\mathcal{D}_s(\theta_0)$ of angles α such that for any generalized parallelogram with angle α the set of points in direction θ_0 which never return parallel to themselves has dimension at most $s \in [0, 1/2]$. We also consider the set $\mathcal{E}_s$ of right triangles for which the set of non-periodic points which are perpendicular to a fixed leg of the triangle has dimension at most $s \in [0, 1/2]$. In Theorem 4.7 we prove that the Hausdorff dimension of $\mathcal{D}_s$ and $\mathcal{E}_s$ are at least $2s$, these sets are residual and have box dimension 1.

The proofs of Theorems 4.6 and 4.7 come from purely number theoretic arguments. The approximation of t by ω is a classical area of research in number theory which is referred to as inhomogeneous Diophantine approximation. A classical result in this direction is the theorem of Minkowski [C] which states that if t is not in the orbit of ω then $\|t + p\omega\| < 1/(4p)$ has infinitely many integer solutions p and the constant $1/4$ can not be improved in general. Here $\|\cdot\|$ is the standard distance on $\mathbb{S}^1$. In the spirit of Minkowski's theorem we consider the set

$$\{t \in \mathbb{S}^1 : \|t + p\omega\| < p^{-\mu} \text{ for infinitely many } p \in \mathbb{N}\}.$$

For any $\mu > 1$ in Theorem 3.2 we prove that the Hausdorff dimension of this set is μ^{-1}. Theorem 4.6 follows easily from a similar statement for approximations along an arithmetic subsequence.

Another classical result in number theory is Jarnik's theorem on the Hausdorff dimension of well approximable irrational numbers [J]. In the spirit of Jarnik's theorem we consider the set

$$\{\omega \in \mathbb{S}^1 : \|t + p\omega\| < p^{-\mu} \text{ for infinitely many } p \in \mathbb{N}\}.$$

For any $\mu > 1$ Levesley has proven that the Hausdorff dimension of this set is $2/(1+\mu)$ [L]. The proof of Theorem 4.7 uses a similar inhomogeneous Jarnik result generalized to arithmetic subsequences.

The structure of the paper is as follows. In Section 2 we describe the needed background results on dimension theory and billiards In Section 3 we state the purely number theoretic results and prove the new results. In Section 4 we state and prove the new billiard results.

Finally we remark that a generalization in a different direction of Cipra *et. al.*'s results was obtained by one of us in [Tr].

2. Preliminaries

2.1. **Dimension.** Let Y be a subset of $\mathbb{R}^n$. Let $N(\epsilon)$ denote the minimal number of ϵ balls needed to cover Y.

Definition 2.1. *For a subset Y of $\mathbb{R}^n$, the lower box dimension of Y, denoted by $\dim_{LB} Y$ is given by*

$$\liminf_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log 1/\epsilon} .$$

The upper box dimension $\dim_{UB}$ is defined similarly, replacing the $\liminf$ by $\limsup$. If $\dim_{UB} Y$ and $\dim_{LB} Y$ both exist and are equal, we define the box dimension of Y to be this value, and write $\dim_B Y = \dim_{UB} Y = \dim_{LB} Y$.

For a subset U of $\mathbb{R}^n$, we let $\operatorname{diam}(U)$ denote the diameter of the set U.

Definition 2.2. *Let $s \in [0, \infty]$. The s-dimensional Hausdorff measure $\mathcal{H}^s(Y)$ of a subset Y of $\mathbb{R}^n$ is defined by the following limit of covering sums:*

$$\mathcal{H}^s(Y) = \lim_{\epsilon \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : Y \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \sup_i \operatorname{diam} U_i \leq \epsilon \right\} \right) .$$

It is easy to see that there exists a unique $s_0 = s_0(Y)$ such that

$$\mathcal{H}^s(Y) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0 \end{cases} . \tag{1}$$

Definition 2.3. *The unique number s_0 given by Equation (1) is defined to be the Hausdorff dimension of Y and is denoted by $\dim_H Y$.*

Standard arguments give that for a subset Y of $\mathbb{R}^n$,

$$\dim_H Y \leq \dim_{LB} Y \leq \dim_{UB} Y$$

There are examples which show that these inequalities may be strict.

The box dimension can also be defined in terms of covering sums. The only change being that the covering sets all have equal diameter. We note that in order to estimate the box dimension, it suffices that the diameters of the covering sets tend to 0 along a geometric sequence.

Lastly, we define the Hausdorff dimension of a measure:

Definition 2.4. *Let μ be a Borel probability measure on X. Then the Hausdorff dimension of the measure μ is defined by*

$$\dim_H \mu = \inf_Y \left\{ \dim_H Y : \mu(Y) = 1 \right\} .$$

We remark that the dimension of a measure is clearly always less than or equal to the dimension of its (Borel) support. There is a well known method of computing lower bounds on the dimension of a measure or a set. Let $\mathcal{U}(x, r)$ denote the ball of radius r centered at x.

Lemma 2.5. *If for some finite measure* μ

$$\liminf_{r\to 0} \frac{\log \mu(\mathcal{U}(x,r))}{\log r} \geq s \text{ on a set of } \mu\text{–positive measures}$$

then the dimension of the measure is at least s.

A survey of the methods and results in dimension theory can be found in [F, P].

2.2. **Directional billiard transformation.** Consider a polygon $Q \subset \mathbb{R}^2$. A billiard ball, i.e. a point mass, moves inside Q with unit speed along a straight line until it reaches the boundary ∂Q, then instantaneously changes direction according to the mirror law: "the angle of incidence is equal to the angle of reflection," and continues along the new line. For the moment we assume that Q is convex. We will describe the billiard map in Q as a transformation of the set X of rays which intersect Q.

We parameterize X via two parameters. The first parameter is the angle θ between the ray and the positive x–axis. To define the second parameter consider the perpendicular cross section X_θ to the set of rays whose angle is θ. The set X_θ is simply an interval, thus the second parameter is then the arc-length on X_θ for a fixed orientation. For a non-convex polygon we must differentiate the portion of rays which enter and leave the polygon several times. The above construction can be done locally, yielding the set X_θ which consists of a finite union of intervals. Let w be the unnormalized length measure on X_θ.

The billiard map $T : X \to X$ takes the ray, which contains a segment of a billiard trajectory oriented by the direction of its motion, to the ray which contains the next segment of this trajectory after the reflection in the boundary.

A polygon Q is called a generalized parallelogram if every side of Q is parallel to one of two fixed directions. We suppose that one of these directions is the direction of the positive x-axis. Then there is a unique $0 < \alpha < \pi/2$ such that the angle between any pair of sides is $0, \alpha$ or $\pi - \alpha$. We suppose throughout that $\alpha \notin \mathbb{Q} \cdot \pi$, that is that Q is not a rational polygon. We remark that the billiard in a right triangle is equivalent to the billiard in a rhombus consisting of four copies of the triangle via the process of unfolding.

The billiard map in a generalized parallelogram has the following special properties (see figure 1). The interval X_θ can be partitioned into three sets U_θ, R_θ and D_θ which consist of a finite union of intervals such that $T^2 U_\theta \subset X_{\theta+2\alpha}$, $T^2 R_\theta \subset X_\theta$ and $T^2 D_\theta \subset X_{\theta-2\alpha}$ and each interval in U_θ, R_θ and D_θ is mapped isometrically onto its image. The length measure w is T–invariant. Furthermore there is a constant K such that

$$w(D_\theta) \leq K|\sin\theta| \quad \text{and} \quad w(U_\theta) \leq K|\sin(\theta-\alpha)|. \tag{2}$$

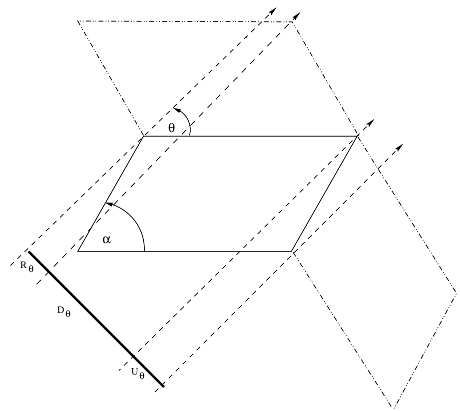

FIGURE 1.

We call the endpoints of the intervals of U_θ and D_θ the singularities of T^2.

Let $Z_n := X_{(\theta+2n\alpha)}$ and $Z := \cup_{n\in\mathbb{Z}} Z_n$. The set Z_n is called the nth level of Z. The set Z is T^2-invariant with infinite invariant measure w.

There are several good survey articles on billiards [G1, G2, T]. The structure discussed here is described in more detail in [GT].

3. Number theoretic results

In this section we work with the circle $\mathbb{R}/\mathbb{Z}$ of unit length rather than the circle of length 2π. The role of θ and α in the billiard sections is played by $t = (\theta \mod \pi)/\pi$ and $\omega = (2\alpha \mod \pi)/\pi$.

3.1. **Inhomogeneous Diophantine approximation.** Let

$$\mathcal{A}_{\mu,\omega} := \{t : \quad \|t + p\omega\| < p^{-\mu} \text{ for infinitely many positive and infinitely many negative } p \in \mathbb{Z}\}.$$

Let $m, l \in \mathbb{N}$ be fixed such that $0 \le l < m$ and let

$$\mathcal{A}_{\mu,\omega}(l,m) := \{t : \|t + p\omega\| < p^{-\mu} \text{ for infinitely many positive and infinitely many negative } p \equiv l \mod m\}.$$

Note that $\mathcal{A}_{\mu,\omega}(l,m) \subset \mathcal{A}_{\mu,\omega}$ and $\mathcal{A}_{\mu,\omega} = \cup_{0\le l<m} \mathcal{A}_{\mu,\omega}(l,m)$.

Minkowski has shown that for t not in the orbit of ω the inequality $\|t + p\omega\| < 1/(4p)$ has infinitely many solutions and in general the constant $1/4$ is optimal [C]. A simple Borel–Cantelli argument tells us that directions which can be approximated better than in the statement of Minkowski's theorem have zero measure.

Proposition 3.1. *Suppose that $\mu > 1$ and $\omega \notin \mathbb{Q}$. The Lebesgue measure of $\mathcal{A}_{\mu,\omega}$ is 0 while $\dim_B \bigcap_{l=0}^{m-1} \mathcal{A}_{\mu,\omega}(l,m) = 1$. The sets $\mathcal{A}_{\mu,\omega}(l,m)$ are residual. Hence, the set $\bigcap_{\mu\in\mathbb{Q},\ \mu>1} \bigcap_{l=0}^{m-1} \mathcal{A}_{\mu,\omega}(l,m)$ is also residual and has box dimension 1.*

Proof. For simplicity, if $0 < a < b$ we denote the interval $(-b,-a)$ by $-(a,b)$. Let $S(l,m) = \{n \in \mathbb{Z} : n \equiv l \mod m\}$, this yields

$$\mathcal{A}_{\mu,\omega}(l,m) = \bigcap_{|k|=1}^{\infty} \bigcup_{p \in S(l,m) \cap \mathrm{sgn(k)}[|\mathrm{k}|\mathrm{m},\infty)} \left(p\omega - \frac{1}{2|p|^{\mu}}, p\omega + \frac{1}{2|p|^{\mu}} \right) \tag{3}$$

and

$$\mathcal{A}_{\mu,\omega} = \bigcap_{|k|=1}^{\infty} \bigcup_{p \in \mathrm{sgn(k)}[|\mathrm{k}|,\infty)} \left(p\omega - \frac{1}{2|p|^{\mu}}, p\omega + \frac{1}{2|p|^{\mu}} \right)$$

thus since $\{p\omega : \ p \in S(l,m) \cap [|k|m,\infty)\}$ and $\{p\omega : \ p \in S(l,m) \cap -[|k|m,\infty)\}$ are dense $\mathcal{A}_{\mu,\omega}(l,m)$ is residual. Thus the set $\bigcap_{k=0}^{m-1} \mathcal{A}_{\mu,\omega}(l,m)$ is dense, hence it's lower and upper box dimension are 1. To see that the Lebesgue measure of $\mathcal{A}_{\mu,\omega}$ is 0 note that the sequence $(2|p|^{-\mu})_{p\in\mathbb{Z}}$ has finite sum for $\mu > 1$. Hence, by the Borel–Cantelli lemma almost no point t is contained in more than a finite number of intervals $(p\omega - 1/2|p|^{\mu}, p\omega + 1/2|p|^{\mu})$. □

Due to this proposition we consider the dimension of the set $\mathcal{A}_{\mu,\omega}$ of t's with better approximations.

Theorem 3.2. *For any $\mu > 1$ and $m \in \mathbb{N}$*

$$\dim_H \bigcap_{l=0}^{m-1} \mathcal{A}_{\mu,\omega}(l,m) = \frac{1}{\mu}$$

Proof. We begin by proving the upper bound. For any $\varepsilon > 0$ Equation (3) implies

$$\mathcal{H}^{\frac{1}{\mu}+\varepsilon}(\mathcal{A}_{\mu,\omega}(l,m)) \leq 2 \times \liminf_{k\to\infty} \sum_{p=k}^{\infty} \left(\frac{1}{p^{\mu}}\right)^{\frac{1}{\mu}+\varepsilon} = 0.$$

We turn to the lower bound. For $n_k \in \mathbb{Z}$ let $\mathcal{J}_k := \mathrm{sgn(n_k)}[|\mathrm{n_k}|, 2|\mathrm{n_k}|]$ and

$$\hat{T}^{\mu}_{n_k} := \bigcup_{qm+k \in \mathcal{J}_k} \left((qm+k)\omega - \frac{1}{2(2|n_k|)^{\mu}}, (qm+k)\omega + \frac{1}{2(2|n_k|)^{\mu}} \right).$$

Given any integer sequence $\{n_k\}$ which meets all residue classes infinitely often for both positive and negative n_k we have that

$$\hat{S}_{\{n_k\}} := \bigcap_{k=1}^{\infty} \hat{T}^{\mu}_{n_k} \subset \bigcap_{l=0}^{m-1} \mathcal{A}_{\mu,\omega}(l,m).$$

To prove the lower bound we will use induction to construct a sequence n_k which is simply a subsequence of the denominators q_m of the continued fraction approximants. Given the sequence $\{n_k\}$ we will construct a measure m supported on the set $\hat{S}_{\{n_k\}}$ whose dimension is μ^{-1}. The idea of our construction is as follows: the faster the sequence n_k increases the better the points $((qm+k)\omega)_{qm+k\in\mathcal{J}_k}$ are distributed along

the circle. If a point x belongs to $\mathcal{A}_{\mu,\omega}$ then it has to be infinitely often in intervals of the form $J_q = (q\omega - |q|^{-\mu}/2, q\omega + |q|^{-\mu}/2)$. So for some large n_k it has to fall into J_q for some $q \in \mathrm{sgn(n_k)}[0, 2|\mathrm{n_k}|$. The set $S_{\{n_k\}}$ forces the point to be in the "right half" of the collection of intervals J_q, where q runs from 0 to $2n_k$. If the sequence $\{n_k\}$ is sufficiently sparse this restriction turns out to be mild enough to maintain the dimension of $\mathcal{A}_{\mu,\omega}$.

Let $n_1 = q_1$ and assume that the sequence n_i is constructed up to $i = k-1$. Then the set $\bigcap_{l=1}^{k-1} \hat{T}^{\mu}_{n_l}$ consists of a finite number of open intervals $\{\hat{I}_i^{k-1}\}$ satisfying

$$\left|\hat{I}_l^{k-1}\right| \le (2|n_{k-1}|)^{-\mu} \tag{4}$$

with equality if and only if they are completely contained in $\cap_{l=1}^{k-2}\hat{T}^{\mu}_{n_l}$. We will drop the hat from the notation for any interval for which equality holds.

We consider the set

$$T^{\mu}_{n_k} := \bigcup_l I_l^k$$

and

$$S_{\{n_k\}} := \bigcap_{k=1}^{\infty} T^{\mu}_{n_k}.$$

Clearly $S_{\{n_k\}} \subset \hat{S}_{\{n_k\}}$.

Assume that inductively $\{n_i\}$ is already constructed for $i \le k-1$. It is well known that for any $k \in \mathbb{N}$ the sequence $\{(pm+k)\omega : \ p \in \mathbb{Z}\}$ is well distributed in $\mathbb{S}^1$, i.e. for any continuous function f and any $\varepsilon > 0$ there is a number N such for any $q \in \mathbb{Z}$

$$\left|\frac{1}{n}\sum_{p=0}^{n-1} f((q+pm+k)\omega) - \int f\right| < \varepsilon$$

for any $n \ge N$. Thus if $k \mod 2m \le m-1$ we can choose n_k to be the least positive $q_r > |n_{k-1}|$ such that the following two conditions hold:

$$\frac{|I_l^{k-1}|}{2} \le \frac{\mathrm{Card}\{pm+k \in \mathcal{J}_k : \ (pm+k)\omega \in I_l^{k-1}\}}{|n_k|} \le 2|I_l^{k-1}| \tag{5}$$

and

$$\frac{\log \prod_{i=1}^{k-1}|n_i|}{\log |n_k|} \to 0. \tag{6}$$

otherwise if $k \mod 2m \ge m$ we choose $n_k = -q_r$ satisfying Conditions (5) and (6). This finishes the construction of the sequence $\{n_k\}$. Clearly, the sequence $\{n_k\}$ meets all residue classes infinitely often for positive and negative n_k.

We next construct a measure on the set $S_{\{n_k\}}$ with the desired dimension. We assume that I^0 consists of the whole circle and thus has

length 1. We begin by a recursive definition of an outer measure m^*. We put $m^*(I^0) := 1$ and

$$m^*(I_l^k) := \frac{m^*(I_j^{k-1})}{\mathrm{Card}\{j :\ I_l^k \subset I_j^{k-1}\}}.$$

We will now compute an upper bound on the outer measure. Using Equations (5) and (4) we have

$$\begin{aligned} m^*(I_j^{k+1}) &\leq \frac{2m^*(I_l^k)}{|I_l^k||n_{k+1}|} \\ &= 2m^*(I_l^k)\frac{(2|n_k|)^\mu}{|n_{k+1}|} \\ &\leq 2^{(1+\mu)k}\prod_{i=1}^{k}\frac{(|n_i|)^\mu}{|n_{i+1}|} \qquad (7) \\ &\leq 2^{(1+\mu)k}\frac{(n_1)^\mu}{|n_{k+1}|}\prod_{i=2}^{k}(|n_i <)^{\mu-1}. \end{aligned}$$

The fast growth rate on n_k assumed by Equation (6) implies that $m(I_j^k) \to 0$ as $k \to \infty$. Thus it is clear that m^* satisfies Kolmogorov's compatibility conditions and can be extended to a measure m on $S_{\{n_k\}}$.

We are now ready to prove that the dimension of the measure is at least $1/\mu$. Equation (7) implies

$$\begin{aligned} \frac{\log m(I_l^{k+1})}{\log |I_l^{k+1}|} &\geq \frac{\log 2^{(1+\mu)k} + \sum_{i=2}^k \log |n_i|^{\mu-1} + \mu \log n_1 - \log |n_{k+1}|}{\log(2|n_{k+1}|)^{-\mu}} \\ &= \frac{-kC_1 - C_2\sum_{i=2}^k \log |n_i| - C_3 + \log |n_{k+1}|}{C_4 + \mu \log |n_{k+1}|} \end{aligned}$$

where the C_i are all positive constants. Equation (6) implies that

$$\lim_{k\to\infty} \frac{\log m(I_l^{k+1})}{\log |I_l^{k+1}|} \geq \frac{1}{\mu}.$$

To use Lemma 2.5 to conclude our result we must evaluate the ratio $\log m(I)/\log |I|$ for the "intermediate" intervals I, that is when the interval I satisfies

$$\frac{1}{(2|n_k|)^\mu} < |I| < \frac{1}{(2|n_{k-1}|)^\mu}.$$

Let $\mathcal{U}_a(B)$ be the neighborhood of radius a around the set B and let $r(I) := \mathrm{Card}\{qm+k :\ qm+k \in \mathcal{J}_k \text{ and } (qm+k)\omega \in I\backslash \mathcal{U}_{(2n_k)^{-\mu}}(\partial I)\}$.

If $r = 1$ then we denote by I_l^k the unique such interval which is contained in I. Then

$$\frac{\log m(I)}{\log |I|} = \frac{\log m(I_l^k)}{\log |I|} > \frac{\log m(I_l^k)}{\log |I_l^k|}.$$

Now suppose $r \geq 2$. Since $|n_k|$ is some continued fraction convergent q_{m_k} we have

$$\min_{|n_k| \leq |p_1| < |p_2| \leq |2n_k|} d(p_1\omega, p_2\omega) \geq \frac{1}{|n_k|+2}. \tag{8}$$

We remark that this is the only place where we use that the n_k is a subsequence of the continued fraction convergents. Equations (8) and (4) imply (assuming that $|n_k| \geq 2$) that

$$|I| \geq (r-1)\frac{1}{|n_k|+2} \geq (r-1)|I_l^k|^{\frac{1}{\mu}}.$$

Thus we can make the following estimate

$$\begin{aligned} \frac{\log m(I)}{\log |I|} &\geq \frac{\log m(I)}{\log((r-1)\cdot |I_l^k|^{\frac{1}{\mu}})} \\ &\geq \frac{\log(r \cdot m(I_l^n))}{\log((r-1)\cdot |I_l^k|^{\frac{1}{\mu}})} \\ &= \frac{\log r + \log m(I_l^k)}{\log(r-1) + \frac{1}{\mu}\log |I_l^k|} \end{aligned} \tag{9}$$

For any $\varepsilon > 0$, using Inequality (9) we can choose $k(\varepsilon)$ sufficiently large such that for any $k \geq k(\varepsilon)$ we have that

$$\frac{\log m(I)}{\log |I|} \geq \frac{\log r + \frac{1}{\mu+\varepsilon}\log |I_l^k|}{\log(r-1) + \frac{1}{\mu}\log |I_l^k|}.$$

Thus

$$\frac{\log m(I)}{\log |I|} \geq \frac{\mu}{\mu+\varepsilon} > \frac{1}{\mu}$$

where the last inequality holds for any sufficiently small ε.

Now Lemma 2.5 implies our result. □

3.2. **Jarnik's theorem.** Let

$$\begin{aligned} \mathcal{B}_{\mu,t} := \{\omega : \|t + p\omega\| < p^{-\mu} &\text{ for infinitely many positive} \\ &\text{and infinitely many negative } p \in \mathbb{Z}\}. \end{aligned}$$

Let $m, l \in \mathbb{N}$ be fixed such that $0 \leq l < m$ and let

$$\begin{aligned} \mathcal{B}_{\mu,t}(l,m) := \{\omega : \|t + p\omega\| < p^{-\mu} &\text{ for infinitely many positive} \\ &\text{and infinitely many negative } p \equiv l \mod m\}. \end{aligned}$$

Then $\mathcal{B}_{\mu,t}(l,m) \subset \mathcal{B}_{\mu,t}$ and $\mathcal{B}_{\mu,t} \supset \bigcup_{0 \leq l < m} \mathcal{B}_{\mu,t}(l,m)$.

Proposition 3.3. *Suppose that $\mu > 1$, and $\omega \notin \mathbb{Q}$. The Lebesgue measure of $\mathcal{B}_{\mu,t}(l,m)$ is 0 while $\dim_B \mathcal{B}_{\mu,t}(l,m) = 1$. The sets $\mathcal{B}_{\mu,t}(l,m)$ are residual. Hence, the set $\bigcap_{\mu \in \mathbb{Q},\ \mu > 1} \bigcap_{0 \leq l < m} \mathcal{B}_{\mu,t}(l,m)$ is also residual and has box dimension 1.*

Proof. Let $S(l,m) = \{n \in \mathbb{N} : n \equiv l \mod m\}$, this yields

(10)
$$\mathcal{B}_{\mu,t}(l,m) = \bigcap_{|k|=1}^{\infty} \bigcup_{p \in S(l,m)\cap \mathrm{sgn(k)}[|\mathrm{k}|,\infty)} \bigcup_{i=0}^{p-1} \left(\frac{t+i}{p} - \frac{1}{2|p|^{\mu+1}}, \frac{t+i}{p} + \frac{1}{2|p|^{\mu+1}} \right)$$

hence $\mathcal{B}_{\mu,t}(l,m)$ is residual. The set $\mathcal{B}_{\mu,t}(l,m)$ is dense, thus it's lower and upper box dimension are 1. We have $\sum 1/2p^{\mu+1} < \infty$, hence, by the Borel–Cantelli lemma almost no point ω is contained in more than a finite number of intervals $(\frac{t+i}{p} - \frac{1}{2|p|^{\mu+1}}, \frac{t+i}{p} + \frac{1}{2|p|^{\mu+1}})$, i.e. the Lebesgue measure of $\mathcal{B}_{\mu,t}(l,m)$ is 0.

□

The following generalization of Jarnik's classical theorem [J] is a special case of a more general result which has been proven by Levesley:

Theorem 3.4. [L] *For any $\mu > 1$*
$$\dim_H \mathcal{B}_{\mu,t} = \frac{2}{1+\mu}.$$

We are going to improve Levesley's Theorem in this special setting. The proof essentially follows his ideas and we do not claim any originality.

Theorem 3.5. *For any $\mu > 1$ and $m \in \mathbb{N}$*
$$\dim_H \bigcap_{l=0}^{m-1} \mathcal{B}_{\mu,t}(l,m) = \frac{2}{1+\mu}.$$

We will need the (adapted to our case) notion of an *ubiquitous* system. For fixed l,m let $R_k(l,m) = \{t \in [0,1) : \|(km+l)t - \omega\| = 0\}$.

Definition 3.6. *Let $\rho\colon \mathbb{N} \to \mathbb{R}$ with $\lim_{N\to\infty} \rho(N) = 0$ be given. Then the family of point sets $\{R_k(l,m)\}$ is said to be* ubiquitous *with respect to ρ if*
$$\lim_{N\to\infty} Leb \left([0,1) \setminus \bigcup_{k=1}^{N} \mathcal{U}(R_k(l,m), \rho(N)) \right) = 0.$$

We can express the sets $\mathcal{B}_{\mu,t}(l,m)$ in the following way
$$\mathcal{B}^{+}_{\mu,t}(l,m) := \bigcap_{K=0}^{\infty} \bigcup_{k=K}^{\infty} \mathcal{U}\left(R_k(l,m), (km+l)^{-\mu-1}\right),$$
$$\mathcal{B}^{-}_{\mu,t}(l,m) := \bigcap_{K=0}^{\infty} \bigcup_{-k=K}^{\infty} \mathcal{U}\left(R_k(l,m), (km+l)^{-\mu-1}\right)$$

and
$$\mathcal{B}_{\mu,t}(l,m) = \mathcal{B}^{+}_{\mu,t}(l,m) \cap \mathcal{B}^{-}_{\mu,t}(l,m).$$

We are going to use the following special case of the more general Theorem 2 in [DRV] proved by Dodson, Rynne and Vickers

Theorem 3.7 ([DRV]). *Suppose that for each $0 \le l < m$ the family $\{R_k(l,m)\}$ is ubiquitous with respect to ρ. Then*

$$\dim_H \bigcap_{l=0}^{m-1} \mathcal{B}_{\mu,t}(l,m) \ge \min\left\{1, \limsup_{N\to\infty} \frac{\log \rho(N)}{\mu+1}\right\}.$$

Levesley proved by using discrepancy estimates that the system $\{\bigcup_{l=0}^{m-1} R_k(l,m)\}$ is ubiquitous with respect to the function $\rho(N) := K\frac{\log^{5+\epsilon} N}{N^2}$. Noting that the sequence $(kmt+l)_k$ has the same discrepancy as the sequence $(kmt)_k$ we can follow the arguments of Levesley and obtain that $\{R_k(l,m)\}$ is ubiquitous with respect to the function $\rho(N) := K\frac{\log^{5+2\varepsilon}(Nm+l)}{(Nm+l)^2}$. Now Theorem 3.7 implies Theorem 3.5.

4. Billiard results

4.1. **Strong recurrence in all directions.** Fix a generalized parallelogram. For every direction θ let F_θ be the set of $x \in X_\theta$ whose forward orbit never returns parallel to x and let $G_\theta := X_\theta \setminus F_\theta$.

Theorem 4.1. *For any generalized parallelogram, for every direction θ, the set F_θ has lower box counting dimension at most 1/2.*

This is an improvement of the main theorem of [GT] which asserts that for every θ the set F_θ has measure 0. We remark that the set G_θ is open and dense for each θ.

Unfolding a right triangle with angle $\alpha/2$ around its right angle yields a rhombus with angle α. We consider a direction θ in the rhombus which is perpendicular to one of the legs of the triangle. The orbits which start in X_θ and return to X_θ, when considered as orbits in the triangle are twice perpendicular to a side. As explained in the introduction, such orbits must be periodic. Thus applying Theorem 4.1 to this direction yields the following improvement of the theorem of Cipra, Hansen and Kolan [CHK]:

Corollary 4.2. *For a right triangle, the set of points which are perpendicular to one of the legs of the triangle and whose orbit is not periodic has lower box counting dimension at most 1/2.*

To prove Theorem 4.1 we start with a lemma which is essentially contained in [GT]

Lemma 4.3. *Fix an interval X_θ and a positive integer N. Then there is a partition of U_θ into $j_N \le CN$ intervals for some positive constant C such the forward orbit of each interval of the partition is an isometric mapping until one of the following happens 1) the orbit of the interval reaches level N before returning to X_θ, or 2) the orbit of the interval*

returns to X_θ without having reached level N. The orbits of all these intervals are mutually disjoint until they possibly return to X_θ.

A similar statement is true for D_θ and level $-M$.

Proof. Consider the set V_N of the singularities of T^2 on levels 0,1 to N. The set V_N has cardinality at most CN (here C can be taken to be twice the number of vertices of the polygon). For each $v \in V_N$ consider the first (if any) preimage which is on level 0. We remark the length of time for $v \in V_N$ to be mapped to level 0 is not necessarily bounded. This forms the indicated partition of U_θ. The mapping T^{2j} is continuous on each element of the partition until it is mapped into level $N+1$ or returns to level 0.

The disjointness follows from the invertability of T^2. To see this suppose that $x, y \in X_\theta$, $x \neq y$ but $T^{2j}x = T^{2k}y$ with $j \geq k$. We can not have $j = k$ since then the point z has two preimages. Since $T^{2(j-k)}x = y$ the orbit of x has returned to X_θ at time $2(j-k) > 0$ and its orbit is disjoint from y's before this time as claimed.

If one of the intervals (call it J) never returned to X_θ or reached level N then the set $\cup_{j\in\mathbb{N}} T^{2j}J$ would consists of an infinite number of disjoint intervals of the same length, yet it would be contained in $\cup_{0\leq n\leq N} Z_n$. This later set consists of a finite union of intervals of finite length yielding the desired contradiction. $\square$

Proof of Theorem 4.1. Fix an arbitrary θ. We will argue about the dimension of $F_\theta \cap D_\theta$. The argument for $F_\theta \cap U_\theta$ is similar with the proviso that since we use Equation (2) $\theta - 2N\alpha$ must be close to 0 for D_θ while for U_θ, $\theta + 2N\alpha$ must be close to α. Because of this proviso, since we need both $F_\theta \cap D_\theta$ and $F_\theta \cap U_\theta$ to be simultaneously small, rather than using the approximants by 2α we will use the approximants by α, using the even iterates for the set $F_\theta \cap D_\theta$ and the odd iterates for the set $F_\theta \cap U_\theta$.

Consider the set F_N of those points $x \in D_\theta$ whose forward orbit reaches $X_{\theta-2(N+1)\alpha}$ before returning to X_θ. We call this event x exiting from level $-N$ (necessarily to level $-(N+1)$). When the orbit of x reaches level $-N$ and is in the process of exiting level $-N$ it must pass through the "gate" $D_{\theta-2N\alpha}$ on level $-N$.

Using Lemma 4.3 clearly at most CN of the intervals exit before returning to X_θ. Hence the set F_N consists of at most CN intervals which we call I_i. When they exit level $-N$ before returning to level 0 the sum of there lengths can not be more than the width of the gate on level $-N$ through which they exit. Here we emphasize that we use the statement in Lemma 4.3 that the exiting orbits are all disjoint. Thus the total length of the intervals exiting level $-N$ is at most the total length of the gate of level $-N$, i.e. at most $K|\sin(\theta - 2N\alpha)|$.

We use $[x]$ to denote the integer part of x. Let $\{a_i\}$ be a sequence of positive numbers and let $n \in \mathbb{N}$. A simple estimate yields (see [KS]):

$$\sum_{k=1}^{n} \left(\left[\frac{n a_k}{\sum_{i=1}^{n} a_i} \right] + 1 \right) \leq 3n \ . \tag{11}$$

The $1/2+\epsilon$ covering sum of $F_\theta \cap D_\theta$ can be estimated by covering F_N by intervals of the "average exiting length" i.e. by intervals of length $K|\sin(\theta - 2N\alpha)|/j_N$.

The total number of intervals of average length which we need to cover is at most

$$\sum_{k=1}^{j_N} \left(\left[\frac{|I_k|(j_N)}{\sum_{i=1}^{j_N} |I_i|} \right] + 1 \right).$$

Using Inequality (11) and the fact that $j_N \leq CN$, we have

$$\begin{aligned} \mathcal{H}^{1/2+\varepsilon}(F_\theta \cap D_\theta) &\leq \sum_{k=1}^{j_N} \left(\left[\frac{|I_k|(j_N)}{\sum_{i=1}^{j_N} |I_i|} \right] + 1 \right) \left(\frac{K(|\sin(\theta - 2N\alpha)|)}{j_N} \right)^{1/2+\varepsilon} \\ &\leq \frac{3 j_N}{(j_N)^{1/2+\varepsilon}} \left(K(|\sin(\theta - 2N\alpha)|\right)^{1/2+\varepsilon} \qquad (12) \\ &\leq L N^{1/2-\varepsilon} \left((\theta - 2N\alpha) \mod \pi \right)^{1/2+\varepsilon} \end{aligned}$$

for some constant L. To get an effective cover we just need to find a sequence (N_k) such that the inhomogeneous Diophantine approximation $\|(\theta - 2N_k\alpha)/\pi\|$ is small. It is well known that for any $\delta > 0$ the inequality

$$\|(\theta - 2p\alpha)/\pi\| < p^{-1+\delta} \tag{13}$$

is solvable for infinitely many $p \in \mathbb{N}$. We remark that this is a much weaker statement than Minkowski's Theorem [C]. Hence, we can find sequences $(N_k)_{k \in \mathbb{N}}$ such that

$$\begin{aligned} \mathcal{H}^{1/2+\varepsilon}(F_\theta \cap D_\theta) &\leq L N_k^{1/2-\varepsilon} \left(N_k^{-1+\delta} \right)^{1/2+\varepsilon} \\ &\leq L N_k^{-2\varepsilon + (\delta/2) + \delta\varepsilon}. \qquad (14) \end{aligned}$$

For $1 > \varepsilon > \delta > 0$ the right–hand–side tends to zero as $k \to \infty$.

By the observation that we covered $F_\theta \cap D_\theta$ by equal length intervals, we have $\dim_{LB}(F_\theta \cap D_\theta) \leq 1/2$.

To estimate the dimension of $F_\theta \cap U_\theta$ we need to use the second part of Equation (2) in Equation (12). With this change it is useful to consider the variable $\theta' := \theta - \alpha$. Noticing that $\theta + 2p\alpha - \alpha = \theta' + 2p\alpha$ and remarking that $\|((\theta' + 2p\alpha)/\pi\| < p^{-1+\delta}$ has infinitely many integer solutions we conclude that $\dim_{LB}(F_\theta \cap U_\theta) \leq 1/2$ and thus that $\dim_{LB}(F_\theta) \leq 1/2$. □

4.2. **Number theory implies stronger recurrence in some directions.** In the previous section we proved a recurrence theorem for all directions θ. For our first results in this section we consider special directions θ which are extremely well approximable. For these directions we can conclude a stronger recurrence result.

Theorem 4.4. *Consider a generalized parallelogram with angle α and let θ be a direction such that $\|(\theta - p\alpha)/\pi\| < p^{-\mu}$ for infinitely many $p \in 2\mathbb{N}$ and $\|(\theta + p\alpha)/\pi\| < p^{-\mu}$ for infinitely many $p \in 2\mathbb{N}+1$. Then the set F_θ has lower box counting dimension at most $1/(\mu+1)$.*

Proof. The proof is essentially identical to the proof of Theorem 4.1. One only needs to change Inequality (13) to the assumption of the theorem and change (14) accordingly. □

We apply the above result to perpendicular orbits in right triangles. Here we assume that the base of the triangle is parallel to the x–axis.

Corollary 4.5. *Fix $\mu > 1$ and consider a right triangle with angle $\alpha/2$, such that the approximations of $\pi/2$ by the orbit of α in the sense of the previous theorem is of order μ. Then the set of points which are perpendicular to the base of the triangle and whose orbit is not periodic has lower box counting dimension at most $1/(\mu+1)$.*

Theorem 4.4 indicates that to get a better result than Theorem 4.1 one needs to consider directions with better approximations, i.e. $\mu > 1$. Let

$$\mathcal{C}_s := \{\theta : \dim_{LB}(F_\theta) \le s\}.$$

Theorem 4.6. *Fix an arbitrary generalized parallelogram. For any $s \in [0, 1/2]$ we have*

$$\dim_H \mathcal{C}_s \ge \frac{s}{1-s},$$

$\mathcal{C}_s$ is residual and has box dimension 1.

Proof. The first statement is an immediate consequence of the previous theorem and Theorem 3.2 with $m = 2$. The second statement is an immediate corollary of Propositions 3.1. □

Next we will fix the direction and vary the generalized polygons. To do this we assume that one of the sides of the polygon is fixed and we measure the direction θ with respect to this fixed direction. We remark that our estimates on F_θ will depend only on the angle α and no other parameters of the generalized parallelogram. Fix a direction θ_0. Consider the sets

$$\begin{aligned}
\mathcal{D}_s &:= \{\alpha \in \mathbb{S}^1 : \dim_{LB}(F_{\theta_0}) < s \text{ for all generalized parallelograms} \\
&\qquad\qquad \text{with angle } \alpha\} \quad \text{and} \\
\mathcal{E}_s &:= \{\alpha \in \mathbb{S}^1 : \dim_{LB}(F_{perp}) < s \text{ for the right triangle} \\
&\qquad\qquad \text{with angle } \alpha/2\}
\end{aligned}$$

where perp is a direction perpendicular to one of the legs of the triangle.

Theorem 4.7. *(a) For* $s \in [0, 1/2]$ *we have* $\dim_H \mathcal{E}_s \geq 2s$*. The set* $\mathcal{E}_s$ *is residual and has box dimension 1.*
(b) For θ_0 *fixed and* $s \in [0, 1/2]$ *we have* $\dim_H \mathcal{D}_s \geq 2s$*. The set* $\mathcal{D}_s$ *is residual and has box dimension 1.*

Proof. The results on Hausdorff dimension are immediate consequences of Theorem 4.4 and Theorem 3.5 with $m = 2$. The residuality and box dimension results are immediate corollaries of Proposition 3.3. □

Acknowledgements We thank Martin Schmoll and Anatole Stepin for useful suggestions.

References

[B] M. Boshernitzan, *Billiards and rational periodic directions in polygons,* Amer. Math. Monthly 99 (1992) 522–529.
[C] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press, Cambridge, 1957.
[CHK] B. Cipra, R. Hanson, and A. Kolan, *Periodic trajectories in right triangle billiards*, Phys. Rev. E 52 (1995) 2066-2071.
[DRV] M. Dodson, B. Rynne, and J. Vickers, *Diophantine approximation and a lower bound for Hausdorff dimension*, Mathematika 37 (1990), 59–73
[F] K. Falconer. *The geometry of fractal sets,* Cambridge University Press, Cambridge, 1985.
[GSV] G. Galperin, A. Stepin and Ya. Vorobets, *Periodic billiard trajectories in polygons: generation mechanisms,* Russian Math. Surveys 47 (1992) 5–80
[G1] E. Gutkin, *Billiard in polygons,* Physica D, 19 (1986) 311-333.
[G2] E. Gutkin, *Billiard in polygons: survey of recent results,* J. Stat. Phys., 83 (1996) 7–26.
[GT] E. Gutkin, and S. Troubetzkoy, *Directional flows and strong recurrence for polygonal billiards,* in Proceedings of the International Congress of Dynamical Systems, Montevideo, Uruguay, F. Ledrappier et. al. eds (1996) 21–45.
[J] V. Jarnik, *Diophantische Approximationen und Hausdorff Maß,* Math. Sbornik, 36 (1929) 371–382.
[KS] B. Kra and J. Schmeling, *Diophantine numbers, dimension and Denjoy maps,* Acta Arithmetica (to appear).
[L] J. Levesley, *A general inhomogeneous Jarnik-Besicovitch theorem* J. Number Theory 71 (1998) 65–80.
[P] Ya. Pesin, *Dimension theory in dynamical systems,* University of Chicago Press, Chicago, 1997.
[R] Th. Ruijgrok, *Periodic orbits in triangular billiards,* Acta Physical Polonica B 22 (1991) 955–981.
[S] W. Schmidt, *Metrical theorems on fractional parts of sequences*, TAMS 110 (1964), 493–518
[T] S. Tabachnikov, *Billiards,* "Panoramas et Syntheses", Soc. Math. France (1995).
[Tr] S. Troubetzkoy, *Recurrence and periodic billiard orbits in polygons,* preprint.

Freie Universität Berlin, FB Mathematik und Informatik, Arnimallee 2-6, D–14195 Berlin and Institut de mathématiques de Luminy, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France
E-mail address: `schmeling@math.fu-berlin.de`

Centre de physique théorique and Institut de mathématiques de Luminy, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France
E-mail address: `troubetz@iml.univ-mrs.fr`
URL: `http://iml.univ-mrs.fr/~troubetz/`